Journal of Applied Mechanics and Technical Physics, Vol. 46, No. 2, pp. 176–184, 2005

SPECIAL CLASS OF SOLUTIONS OF THE KINETIC EQUATION OF A BUBBLY FLUID

G. Russo,[1] V. M. Teshukov,[2] and A. A. Chesnokov[2]

UDC 532.591+517.948

A new class of solutions is constructed for the kinetic model of bubble motion in a perfect fluid proposed by Russo and Smereka. These solutions are characterized by a linear relationship between the Riemann integral invariants. Using the expressions following from this relationship, the construction of solutions in the special class is reduced to the integration of a hyperbolic system of two differential equations with two independent variables. Exact solutions in the class of simple waves are obtained, and their physical interpretation is given.

Key words: *bubbly fluid, kinetic equation, hyperbolicity, exact solutions.*

Introduction. A kinetic approach has been developed [1–5] to simulate the motion of gas bubbles in a perfect fluid taking into account the effects of their collective interaction. The construction of the equations of motion is based on the calculation of the fluid kinetic energy, which is represented as the quadratic form of the bubble velocity [6], whose coefficients are determined by the fluid flow potential in the region between bubbles. Assuming that all bubbles are identical and are solid imponderable spheres, Russo and Smereka [3] approximately calculated the kinetic energy and Hamiltonian of bubble motion by asymptotic expansion of the solution of the Laplace equation in a small parameter (the ratio of the bubble radius to the average separation between the bubbles). In [3], the kinetic equation describing the evolution of the one-particle distribution was obtained using a system of Hamilton's ordinary differential equations for the bubble coordinates and momenta and a method of deriving Vlasov's equation.

The characteristic properties of the Russo–Smereka kinetic model [3] for the case of one space variable are studied in [7], where hyperbolicity conditions are formulated and Riemann integral invariants and infinite series of conservation laws are found. The exact solutions of the kinetic model in the class of simple waves propagating over a specified, spatially homogeneous, stationary background are obtained in [7, 8].

In the present paper, we propose a new method for constructing solutions of integrodifferential equations that admit a formulation in terms of Riemann invariants. This method is used to find a wide class of solutions of the Russo–Smereka kinetic equation that is described by a system of first-order hyperbolic differential equations.

1. Mathematical Model. In the one-dimensional case, the Russo–Smereka kinetic equation in dimensionless variables is written as follows [3, 7]:

$$f_t^1 + (p-j)f_x^1 + pj_x f_p^1 = 0, \qquad j(t,x) = \int\limits_{-\infty}^{\infty} pf^1\,dp. \tag{1}$$

Here $f^1(t,x,p)$ is the one-particle distribution function for the number of bubbles in the phase space (x,p) of coordinates and momenta, t is time, and j is the first moment of the distribution function. It is assumed that f^1 decreases rapidly at infinity or is finite over the variable p.

[1]University of Catania, Catania 95125, Italy. [2]Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; chesnokov@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 46, No. 2, pp. 33–43, March–April, 2005. Original article submitted August 25, 2004.

In the present paper, we consider the solutions of Eq. (1) in the class of functions that are piecewise continuous in the variable p and have a limited carrier:

$$f^1(t,x,p) = f(t,x,p)(\theta(p-p_1(t,x)) - \theta(p-p_2(t,x))). \tag{2}$$

Here θ is a Heaviside function, $p_1(t,x)$ and $p_2(t,x)$ are the boundaries of the interval in the variable p, outside which the distribution function $f^1(t,x,p)$ is identically equal to zero, and $f(t,x,p)$ is a nonnegative function which is continuously differentiable on the set $\{(t,x,p) \mid t \geq 0,\ x \in \mathbb{R},\ p \in (p_1,p_2)\}$. Substituting the representation of the solution (2) into the kinetic equation (1), we obtain the following integrodifferential system for the quantities $f(t,x,p)$, $p_1(t,x)$, and $p_2(t,x)$:

$$\frac{\partial f}{\partial t} + (p-j)\,\frac{\partial f}{\partial x} + p\,\frac{\partial j}{\partial x}\,\frac{\partial f}{\partial p} = 0 \quad (x \in \mathbb{R},\ p \in (p_1,p_2)), \qquad j(t,x) = \int\limits_{p_1}^{p_2} pf\,dp; \tag{3}$$

$$\frac{\partial p_1}{\partial t} + (p_1 - j)\,\frac{\partial p_1}{\partial x} - p_1\,\frac{\partial j}{\partial x} = 0, \qquad \frac{\partial p_2}{\partial t} + (p_2 - j)\,\frac{\partial p_2}{\partial x} - p_2\,\frac{\partial j}{\partial x} = 0. \tag{4}$$

The goal of this work is to develop methods for constructing solutions of integrodifferential equations and to obtain and analyze a new class of solutions of Eqs. (3) and (4).

We note that in the particular case where $f = f_0 = \text{const}\ \ [x \in \mathbb{R},\ p \in (p_1,p_2)]$, the solution of system (3), (4) reduces to the integration of the two differential equations

$$\begin{aligned} &\frac{\partial p_1}{\partial t} + \Big(p_1 + f_0\,\frac{3p_1^2 - p_2^2}{2}\Big)\frac{\partial p_1}{\partial x} - f_0 p_1 p_2\,\frac{\partial p_2}{\partial x} = 0, \\ &\frac{\partial p_1}{\partial t} + f_0 p_1 p_2\,\frac{\partial p_1}{\partial x} + \Big(p_2 + f_0\,\frac{p_1^2 - 3p_2^2}{2}\Big)\frac{\partial p_2}{\partial x} = 0. \end{aligned} \tag{5}$$

The solutions of the kinetic equations (1) with a piecewise constant distribution function [including step-type distributions that can be reduced to Eqs. (5)] are studied in [9].

Below, we shall consider the class of solutions with a nontrivial distribution function $f(t,x,p) \neq \text{const}$, which also reduces to integrating a system of two differential equations.

2. Riemann Invariants. Using the lemma given below, we prove the existence of a function $R(t,x,p)$ that, like the distribution function $f(t,x,p)$, is conserved along the trajectories

$$\frac{dx}{dt} = p - j, \qquad \frac{dp}{dt} = pj_x \tag{6}$$

on the solutions of the kinetic equation (3).

Lemma 1. *Let $f(t,x,p)$, $p_1(t,x)$, $p_2(t,x)$, $j(t,x)$, and $n(t,x)$ be arbitrary differentiable functions and let the derivatives f_t, f_x, and f_p satisfy Hölder's condition in the variable p. Then, the following equality is valid:*

$$D\Big(\frac{n-1}{p} + \int\limits_{p_1}^{p_2} \frac{f'\,dp'}{p'-p}\Big) = \frac{n_t + ((1-n)j)_x}{p} + \int\limits_{p_1}^{p_2} \frac{D'f'\,dp'}{p'-p}$$

$$+ \frac{(p_{2t} + (p_2 - j)p_{2x} - p_2 j_x)f_2}{p_2 - p} - \frac{(p_{1t} + (p_1 - j)p_{1x} - p_1 j_x)f_1}{p_1 - p} + \frac{\partial}{\partial x}\Big(n - \int\limits_{p_1}^{p_2} f'\,dp'\Big). \tag{7}$$

Here we use the notation of the functions $f = f(t,x,p)$, $f' = f(t,x,p')$, and $f_i = f(t,x,p_i)$ and the operators $D = \partial_t + (p-j)\,\partial_x + pj_x\,\partial_p$, and $D' = \partial_t + (p'-j)\,\partial_x + p'j_x\,\partial_{p'}$.

The proof is performed by direct calculations using the formulas for differentiating integrals with variable limits and the formulas for differentiating singular integrals [10]:

$$\frac{\partial}{\partial \xi} \int\limits_a^b \frac{\varphi(\tau)\,d\tau}{\tau - \xi} = \frac{\varphi(a)}{a-\xi} - \frac{\varphi(b)}{b-\xi} + \int\limits_a^b \frac{\varphi'(\tau)\,d\tau}{\tau - \xi}.$$

We introduce the notation

$$R(t,x,p) = \frac{n(t,x)-1}{p} + \int\limits_{p_1}^{p_2} \frac{f(t,x,p')\,dp'}{p'-p}, \qquad n(t,x) = \int\limits_{p_1}^{p_2} f(t,x,p)\,dp. \tag{8}$$

Note that a consequence of system (3), (4) is the equality

$$n_t + ((1-n)j)_x = \int\limits_{p_1}^{p_2} (f_t + (p-j)f_x + pj_x f_p)\,dp + (p_{2t} + (p_2-j)p_{2x} - p_2 j_x)f_2 - (p_{1t} + (p_1-j)p_{1x} - p_1 j_x)f_1. \tag{9}$$

Thus, if the functions $f(t,x,p)$, $p_1(t,x)$, and $p_2(t,x)$ are a solution of system (3), (4), then, by virtue of Eqs. (7), (8), and (9), the quantity $R(t,x,p)$ satisfies the equation

$$R_t + (p-j)R_x + pj_x R_p = 0.$$

The functions f and R that are conserved along the trajectories (6) will be called Riemann invariants. In greater detail, the characteristic properties of the Russo–Smereka kinetic equation and its reduction to Riemann integral invariants using Eulerian–Lagrangian coordinates are considered in [7].

3. Class of Solutions with Functionally Dependent Riemann Invariants. We consider solutions with a functional relationship between the quantities $f(t,x,p)$ and $R(t,x,p)$. Let $f = \psi(R)$, where ψ is an arbitrary differentiable function. Then, by virtue of (8), for $f(t,x,p) = \bar{f}(p_1,p_2,p)$, we have the nonlinear singular integral equation

$$f = \psi\Bigl(\frac{n-1}{p} + \int\limits_{p_1}^{p_2} \frac{f'\,dp'}{p'-p}\Bigr). \tag{10}$$

Next, we assume the validity of the inequalities

$$0 < p_1 \le p \le p_2 < \infty.$$

If Eq. (10) is solved and the function $\bar{f}(p_1,p_2,p)$ is found, then, calculating the first moment of the function

$$j(t,x) = \bar{j}(p_1,p_2) = \int\limits_{p_1}^{p_2} p\bar{f}(p_1,p_2,p)\,dp$$

and substituting it into (4), we obtain the following closed system of differential equations for the unknowns $p_1(t,x)$ and $p_2(t,x)$:

$$p_{1t} + (p_1-\bar{j})p_{1x} - p_1(\bar{j}(p_1,p_2))_x = 0, \qquad p_{2t} + (p_2-\bar{j})p_{2x} - p_2(\bar{j}(p_1,p_2))_x = 0. \tag{11}$$

Let us show that the function $f(t,x,p)$ satisfies Eq. (3) if p_1 and p_2 are a solution of system (11). Applying the differential operator $D = \partial_t + (p-j)\partial_x + pj_x\partial_p$ to both sides of equality (10), we obtain

$$Df = \psi'_R D\Bigl(\frac{n-1}{p} + \int\limits_{p_1}^{p_2} \frac{f'\,dp'}{p'-p}\Bigr).$$

Since p_1 and p_2 satisfy Eqs. (11), using (7) and (9), the last relation is reduced to the homogeneous singular integral equation

$$\varphi - \psi'_R \int\limits_{p_1}^{p_2} \frac{\varphi'\,dp'}{p'-p} = 0, \tag{12}$$

where $\varphi = pDf$, $D' = \partial_t + (p'-j)\partial_x + p'j_x\partial_{p'}$, and $\varphi' = p'D'f'$. Equation (3) is satisfied if (12) has only the trivial solution $\varphi = pDf = 0$. The uniqueness conditions for the solution of Eq. (12) are formulated in [11]. Verification of these conditions completes the construction of the special class of solutions of the Russo–Smereka kinetic model described by system (11) with functionally dependent Riemann invariants.

4. Solutions with Linearly Dependent Riemann Invariants. We obtain the class of solutions of the integrodifferential system (3), (4) for the case of a linear relationship between the invariants f and R. Let

$$f = \psi(R) = a(R - b), \tag{13}$$

where a and b are constants. Without loss of generality, we set $a^{-1} = -\pi \cot \mu\pi$ ($0 < \mu < 1$ and $\mu \neq 1/2$). Then, for the function f, the following linear singular integral equation arises:

$$f\pi \cot \mu\pi + \int_{p_1}^{p_2} \frac{f'\, dp'}{p' - p} = \frac{1-n}{p} + b. \tag{14}$$

According to the general theory of singular integral equations [11], Eq. (14) is uniquely solvable in the class of functions that satisfy Hölder's condition at the internal points of the interval (p_0, p_1), and are bounded at one of the ends of this interval, and unbounded at the other end.

Employing the methods developed in [11] and tabulated formulas for singular integrals (see [12])

$$\int_{p_1}^{p_2} \Big(\frac{p' - p_1}{p_2 - p'}\Big)^{\mu} \frac{dp'}{p' - p} = \frac{\pi}{\sin \mu\pi} - \pi \cot \mu\pi \Big(\frac{p - p_1}{p_2 - p}\Big)^{\mu},$$

$$\int_{p_1}^{p_2} \frac{1}{p'}\Big(\frac{p' - p_1}{p_2 - p'}\Big)^{\mu} \frac{dp'}{p' - p} = \frac{\pi}{\sin \mu\pi}\Big(\frac{p_1}{p_2}\Big)^{\mu}\frac{1}{p} - \frac{\pi \cot \mu\pi}{p}\Big(\frac{p - p_1}{p_2 - p}\Big)^{\mu},$$

we find the solution of Eq. (14) in explicit form

$$f = \frac{\sin \mu\pi}{\pi}\Big(\frac{1}{p}\Big(\frac{p_2}{p_1}\Big)^{\mu}(1 - n) + b\Big)\Big(\frac{p - p_1}{p_2 - p}\Big)^{\mu} \tag{15}$$

(we chose the class of solutions bounded at the point $p = p_1$ and unbounded at the point $p = p_2$).

Integration of (15) over p yields the following expression for the bubble concentration:

$$n(t, x) = \bar{n}(p_1, p_2) = 1 - (p_1/p_2)^{\mu}(1 - \mu b(p_2 - p_1)). \tag{16}$$

Substitution of (16) into (15) yields the resulting expression for the distribution function in the special class of solutions characterized by the linear relationship (13) between the Riemann invariants f and R:

$$f = \frac{\sin \mu\pi}{\pi}\Big(\frac{1 - \mu b(p_2 - p_1)}{p} + b\Big)\Big(\frac{p - p_1}{p_2 - p}\Big)^{\mu}. \tag{17}$$

Using the obtained distribution function f, we calculate its first moment $j(t, x)$

$$j = \bar{j}(p_1, p_2) = \mu(p_2 - p_1)(1 + (1 + \mu)bp_1/2 + (1 - \mu)bp_2/2) \tag{18}$$

and reduce the solution of the integrodifferential system (3), (4) in the special class to the integration of the two differential equations

$$\frac{\partial p_1}{\partial t} + \Big((1 + \mu + \mu b p_1)p_1 - \mu(p_2 - p_1)\Big(1 + b\,\frac{p_1 + p_2}{2} + \mu b\,\frac{3p_1 - p_2}{2}\Big)\Big)\frac{\partial p_1}{\partial x} - \mu p_1\Big(1 - \mu b(p_2 - p_1) + bp_2\Big)\frac{\partial p_2}{\partial x} = 0,$$

$$\frac{\partial p_2}{\partial t} + \mu p_2\Big(1 - \mu b(p_2 - p_1) + bp_1\Big)\frac{\partial p_1}{\partial x} \tag{19}$$

$$+ \Big((1 - \mu - \mu b p_2)p_2 - \mu(p_2 - p_1)\Big(1 + b\,\frac{p_1 + p_2}{2} + \mu b\,\frac{p_1 - 3p_2}{2}\Big)\Big)\frac{\partial p_2}{\partial x} = 0.$$

The propagation velocities of the characteristics $dx/dt = k = k_{1,2}$ of system (19) are found from the quadratic equation

$$(k + j)^2 - (1 - \mu b(p_2 - p_1))((1 + \mu)p_1 + (1 - \mu)p_2)(k + j) + (1 - \mu b(p_2 - p_1))p_1 p_2 = 0. \tag{20}$$

Here the quantity j is defined by formula (18). Equations (19) are hyperbolic if the following inequality is satisfied:

$$((1 + \mu)p_1 + (1 - \mu)p_2)^2 > 4p_1 p_2/(1 - \mu b(p_2 - p_1)). \tag{21}$$

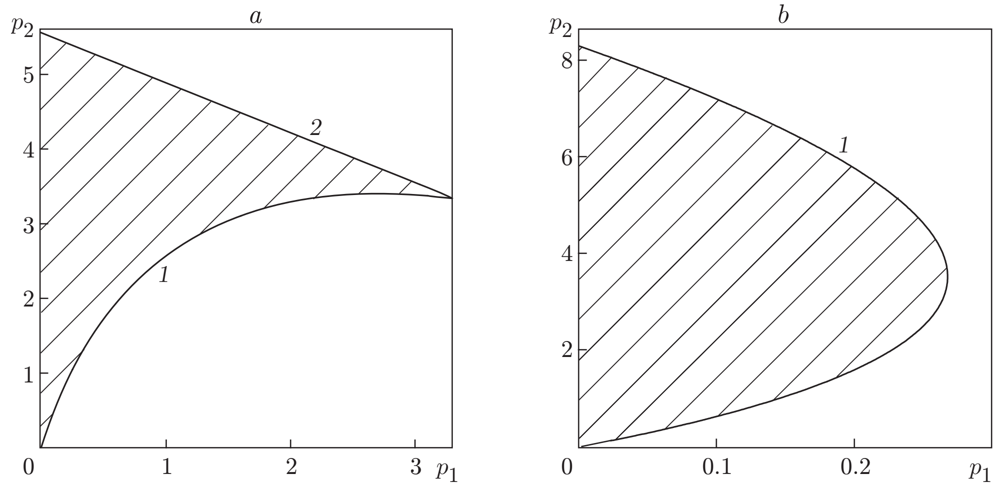

Fig. 1

The requirement of nonnegativity of f and the constraint on the bubble concentration $n < 1$ corresponding to the applicability of the kinetic model (1) (see [3, 7]) lead to the inequality

$$(\theta(b) - 1)bp_2 < 1 - \mu b(p_2 - p_1). \tag{22}$$

The regions in which inequalities (21) and (22) are satisfied for $\mu = 0.4$ and $b = \mp 0.3$ are shown by hatching in Fig. 1. In the case of negative values of b (Fig. 1a), the shaded region is bounded by curve 1 [a consequence of inequality (21)], the straight line $p_1 = 0$, and straight line 2 [corresponds to constraint (22)]. The condition of nonnegativity of the distribution function (22) is satisfied automatically for $b \geq 0$. The region in which the hyperbolicity conditions are satisfied for positive values of the parameter b is shown by hatching in Fig. 1b. The hyperbolicity region is bounded by the straight line $p_1 = 0$ and curve 1, at whose points (21) becomes equality.

We have shown that the special class of solutions is described by the system of differential equations (19) dependent on the parameters μ and b. The system is considerably simplified for $b = 0$. In this case, Eqs. (19) become

$$\begin{aligned} &\frac{\partial p_1}{\partial t} + ((1 + 2\mu)p_1 - \mu p_2)\,\frac{\partial p_1}{\partial x} - \mu p_1\,\frac{\partial p_2}{\partial x} = 0, \\ &\frac{\partial p_2}{\partial t} + \mu p_2\,\frac{\partial p_1}{\partial x} + (\mu p_1 + (1 - 2\mu)p_2)\,\frac{\partial p_2}{\partial x} = 0. \end{aligned} \tag{23}$$

The propagation velocities of the characteristics of Eqs. (23) are given by the formulas

$$k_{1,2} = \frac{1 + 3\mu}{2}\,p_1 + \frac{1 - 3\mu}{2}\,p_2 \mp \frac{1}{2}\sqrt{(1 + \mu)^2 p_1^2 + (1 - \mu)^2 p_2^2 - 2(1 + \mu^2)p_1 p_2}$$

and the hyperbolicity condition (21) has the form

$$(1 - \mu)^2 p_2 > (1 + \mu)^2 p_1.$$

We note that inequality (22) is satisfied automatically.

Let us reduce the differential equations (19) to Riemann invariants. For the propagation velocity of the characteristics of the kinetic equation (1), Teshukov [7] obtained the equation

$$\chi(k + j) = 1 - n + (k + j)\int\limits_{p_1}^{p_2} \frac{f\,dp}{(p - j - k)^2} = 0. \tag{24}$$

Of interest are solutions in which the inequalities $0 < n < 1$ and $f \geq 0$ are satisfied. In this case, the characteristic equation does not have real roots k outside the interval $[p_1 - j, p_2 - j]$. However, the previous results in the special class of solutions show that if the equations are hyperbolic, real roots k_1 and k_2 exist inside this interval: $p_1 - j < k_1 < k_2 < p_2 - j$. To give meaning to Eq. (24) in the interval in which the quantity $p - j - k$ vanishes, we continue the function χ to the complex region and calculate the limiting values of the function of the complex argument $\chi(z)$ from the upper and lower semi-planes on the segment $[p_1, p_2]$:

$$\chi^{\pm}(p) = 1 - n + p^2\Big(\frac{f_1}{p_1 - p} - \frac{f_2}{p_2 - p} + \int\limits_{p_1}^{p_2} \frac{f'_p\,dp'}{p' - p}\Big) \pm \pi i p^2 f_p.$$

Direct calculations show that at the points

$$p = k_{1,2} + j = \frac{1 - \mu b(p_2 - p_1)}{2}\Big((1+\mu)p_1 + (1-\mu)p_2 \mp \sqrt{((1+\mu)p_1 + (1-\mu)p_2)^2 - \frac{4p_1p_2}{1 - \mu b(p_2 - p_1)}}\,\Big)$$

[k_i are roots of the quadratic equation (20)], the real and imaginary parts of the complex functions $\chi^{\pm}(p)$ vanish. We note that the condition $\operatorname{Im}\{\chi(k_i + j)\} = 0$ is equivalent to the equality $f_p = 0$ for $p = k_i + j$. Therefore, at the indicated points, the characteristic equation (24) is satisfied in the sense of the principal value.

According to (8), the Riemann invariants corresponding to the characteristic values $p = k_i + j$ $(i = 1, 2)$ are defined by the formulas

$$r_i = \frac{n - 1}{k_i + j} + \int\limits_{p_1}^{p_2} \frac{f\,dp}{p - j - k_i}.$$

Taking into account (16) and (17), after simple calculations we obtain

$$r_i = b - \cos\mu\pi\Big(b + \frac{1 - \mu b(p_2 - p_1)}{k_i + j}\Big)\Big(\frac{k_i + j - p_1}{p_2 - j - k_i}\Big)^{\mu}. \tag{25}$$

Thus, the system of differential equations (19), which describes the special class of solutions, is reduced to the Riemann invariants

$$\frac{\partial r_1}{\partial t} + k_1\frac{\partial r_1}{\partial x} = 0, \qquad \frac{\partial r_2}{\partial t} + k_2\frac{\partial r_2}{\partial x} = 0.$$

5. Simple Waves. Solutions of the form $f(t, x, p) = \hat{f}(k(t, x), p)$ and $p_i(t, x) = \hat{p}_i(k(t, x))$, where $k(t, x)$ is an arbitrary smooth function, will be called the simple waves of the system of integrodifferential equations (3) and (4). Let us construct a simple wave for the special class of the solutions. This problem reduces to finding solutions of the form $p_i(t, x) = \hat{p}_i(k(t, x))$ to system (19). From the common properties of simple waves, it follows that one of the Riemann invariants (25) remains constant in the wave region. Let a wave propagate at a characteristic velocity $k = k_1$ $(k_1 < k_2)$; then, the Riemann invariant $r_2 = \text{const}$ in the simple wave. Obtaining this solution reduces to determining the functions p_1 and p_2 from the final relations

$$r_2(p_1, p_2) = r_{20} = \text{const}, \qquad k_1(p_1, p_2) = k \tag{26}$$

and integrating the equation

$$k_t + kk_x = 0.$$

For simplicity and illustration, we consider the special class of solutions for the case $b = 0$ [system (23)] and construct a self-similar simple wave that corresponds to the choice $k(t, x) = x/t$. Introducing the notation

$$h = p_2 + p_1, \qquad l = p_2 - p_1, \qquad \xi = x/t \tag{27}$$

and using (27), we bring Eqs. (26) to the form

$$h - 3\mu l - \alpha = 2\xi, \qquad \alpha = l\sqrt{1 + \mu^2 - 2\mu h/l}\,; \tag{28}$$

$$\frac{1}{h - \mu l + \alpha}\Big(\frac{(1-\mu)l + \alpha}{(1+\mu)l - \alpha}\Big)^{\mu} = -r'_{20} \qquad \Big(r'_{20} = \frac{r_{20}}{2\cos\mu\pi}\Big). \tag{29}$$

From formulas (28), α and h can be expressed as functions of l and ξ:

$$\alpha(l, \xi) = -\mu l + l\sqrt{1 - 4\mu^2 - 4\mu\xi/l}\,, \quad h(l, \xi) = 2\xi + 2\mu l + l\sqrt{1 - 4\mu^2 - 4\mu\xi/l}\,.$$

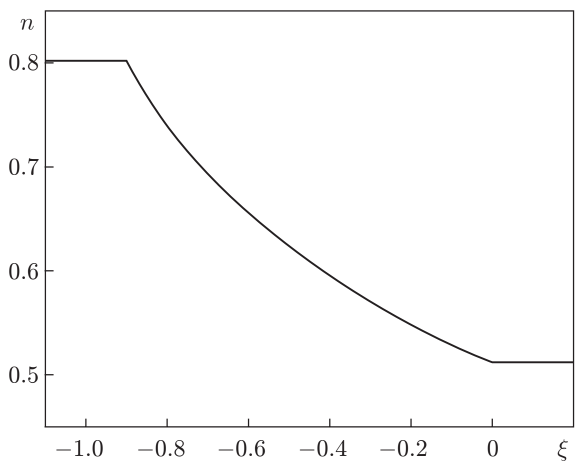

Fig. 2

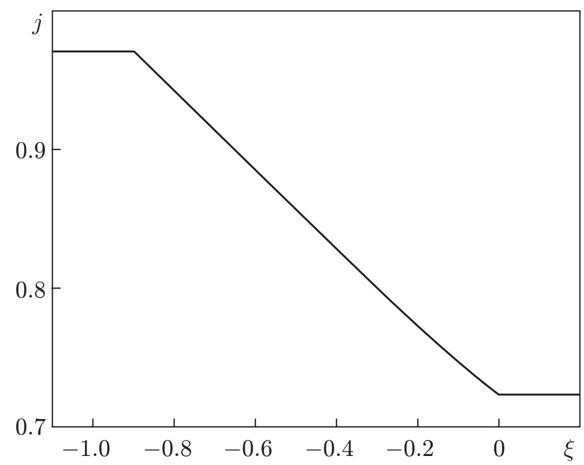

Fig. 3

Then, formula (29) is the closing relation for the determination of the relation $l = l(\xi)$ from the equation

$$F(l,\xi) = \Big(\frac{(1-\mu)l + \alpha(l,\xi)}{(1+\mu)l - \alpha(l,\xi)}\Big)^{\mu} + r'_{20}(\xi + \mu l + \alpha) = 0. \tag{30}$$

The results of an analysis of the unique solvability of the equation $F(l,\xi) = 0$ are given below.

For any ξ from the interval (ξ_1, ξ_2), the equation $F(\xi, l) = 0$ has a unique root $l > 0$ if one of the following conditions is satisfied:

1) $0 < \mu \le \dfrac{1}{\sqrt{5}}, \quad r'_{20} < 0, \quad \xi_1 = \dfrac{1}{r'_{20}}\Big(\dfrac{\mu}{1-\mu}\Big)^{1-\mu}, \quad \xi_2 = 0;$

2) $\dfrac{1}{\sqrt{5}} < \mu < 1, \quad r'_{20} < 0, \quad \xi_1 = \dfrac{1}{r'_{20}}\Big(\dfrac{\mu}{1-\mu}\Big)^{1-\mu}, \quad \xi_2 = \dfrac{5\mu^2 - 1}{r'_{20}(1-\mu^2)}\Big(\dfrac{1-\mu}{1+\mu}\Big)^{\mu};$

3) $0 < \mu < \dfrac{1}{3}, \quad r'_{20} < 0, \quad \xi_1 = 0, \quad \xi_2 = -\dfrac{1}{2r'_{20}}\Big(\dfrac{1-2\mu}{\mu}\Big)^{\mu};$

4) $\dfrac{1}{3} \le \mu < \dfrac{1}{\sqrt{5}}, \quad r'_{20} < 0, \quad \xi_1 = 0, \quad \xi_2 = -\dfrac{1 - 5\mu^2}{r'_{20}(1-\mu^2)}\Big(\dfrac{1-\mu}{1+\mu}\Big)^{\mu}.$

.

For $\xi \in (\xi_1, \xi_2)$, the above conditions guarantee the validity of the inequalities

$$F(l_1,\xi) > 0, \qquad F(l_2,\xi) < 0, \qquad F_{ll} > 0, \tag{31}$$

which allow one to uniquely solve the closing equation (30) and determine the function $l = l(\xi)$ in the interval (ξ_1, ξ_2). However, the relation $l = l(\xi)$ is difficult to write in explicit form; therefore, to solve the equation $F(l,\xi) = 0$ numerically, we use the Newton iterative process $l_{i+1}(\xi) = l_i(\xi) - F(l_i,\xi)/F'_l(l_i,\xi)$, whose convergence is guaranteed by inequalities (31).

From the known functions $l(\xi)$ and $h(\xi)$, we obtain the quantities $p_1(\xi) = (h-l)/2$ and $p_2(\xi) = (h+l)/2$. According to (17), the distribution function for $b = 0$ has the form

$$f = \frac{\sin \mu\pi}{\pi p}\Big(\frac{p - p_1}{p_2 - p}\Big)^{\mu}.$$

As a result, we constructed a particular solution of the kinetic equations (3) and (4).

As an example, we consider a self-similar simple wave with parameters $\mu = 0.4$, $b = 0$, and $r'_{20} = -(\mu/(1-\mu))^{1-\mu} \approx -0.784$ which is defined in the interval $\xi \in (a_1, a_2)$ and adjoin the spatially homogeneous, stationary

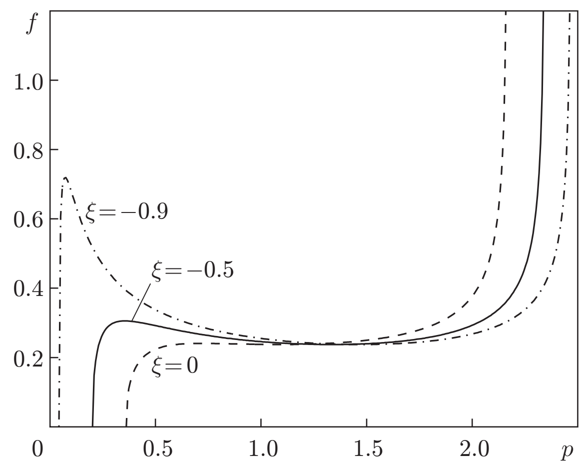

Fig. 4

distributions $f_i(p) = \hat{f}(a_i, p)$ along the characteristics $\xi = a_i$ $(i = 1, 2)$. A solution of the simple-wave type defined by relation (26) exists in any interval $(a_1, a_2) \subset (\xi_1, \xi_2)$ in which

$$\xi_1 = \frac{1}{r'_{20}}\Big(\frac{\mu}{1-\mu}\Big)^{1-\mu} = -1, \qquad \xi_2 = -\frac{1-5\mu^2}{r'_{20}(1-\mu^2)}\Big(\frac{1-\mu}{1+\mu}\Big)^{\mu} \approx 0.2164.$$

This follows from the above analysis of the unique solvability of the closing equation (30) (see conditions 1 and 4, which correspond to this choice of the parameters μ and r'_{20}). For determinacy, let $a_1 = -0.9$ and $a_2 = 0$. As follows from (16) and (18), the bubble concentration in the fluid and the first moment of the distribution function have the form $n = 1 - (p_1/p_2)^\mu$ and $j = \mu(p_2 - p_1)$, respectively. Plots of these functions are given in Figs. 2 and 3. As the self-similar variable ξ decreases, the bubble concentration n increases. In this case, the moment j, which is proportional to the differences p_2 and p_1, also increases, i.e., the distribution-function carrier is extended. Plots of the function $f = \hat{f}(\xi, p)$ for fixed values of ξ are given in Fig. 4. The dashed and dot-and-dashed curves show the distributions $f_i(p)$ to which the simple wave $f = \hat{f}(\xi, p)$ is continuously adjacent along the characteristics $\xi = a_i$. The distribution function $\hat{f}(\xi, p)$ vanishes [at the points $(\xi, p_1(\xi))$ and tends to infinity as $(\xi, p) \to (\xi, p_2(\xi))$. At the internal points of the interval $(p_1(\xi), p_2(\xi))$, the derivative $\hat{f}_p(\xi, p)$ vanishes [at the points $p_{\max} = k_1 + j(\xi) = \xi + j(\xi)$ and $p_{\min} = k_2 + j(\xi)$]. We note that $p_1(\xi) \to +0$ and $n(\xi) \to 1 - 0$ as $\xi \to \xi_1 + 0$. At the point $(\xi, p_{\max}(\xi))$, the values of the function $\hat{f}(\xi, p)$ increase with decrease in the self-similar variable ξ.

In a coordinate system moving together with the wave, the particle trajectory is defined, according to (6), by the following system of ordinary differential equations:

$$\frac{d\xi}{dt} = \frac{p - j - \xi}{t}, \qquad \frac{dp}{dt} = \frac{pj'(\xi)}{t}. \tag{32}$$

Figure 5 gives the results of numerical integration of system (32) in parametric form on the plane of variables (p, ξ) (the arrows show the direction of the particle motion). Bubbles that penetrate into the simple-wave region through the front $\xi = a_1$ and have small momenta $[p_1(a_1) < p < p_* \approx 1.25]$ perform rotational motion in the wave and return to the line $\xi = a_1$. Such particle behavior is due to the fact that the quantity $p - j - \xi$, which determines the sign of the derivative $\xi'(t)$, changes sign in the simple-wave region. This change in sign takes place on the line $p = p_{\max}(\xi) = \xi + j(\xi)$, at whose points, $\hat{f}_p = 0$. The constructed solution describes motion with a critical layer since on the line $p = p_{\max}(\xi)$ in the simple-wave region, the particle velocity $p - j$ coincides with the local velocity of the wave propagating at speed ξ. Bubbles that at $\xi = a_1$ have momenta $p > p_*$ leave the wave region through the front $\xi = a_2$. Particles with momenta $p_1(a_2) < p < j(a_2) + a_2$ [for $a_2 = 0$, $p_1(a_2) \approx 0.3612$ and $j(a_2) \approx 0.7233$] penetrate into the simple-wave region through the characteristic $\xi = a_2$ and leave it through the front $\xi = a_1$. We note that all bubbles that arrive at the line $\xi = a_1$ have close momenta $p \in (p_1(a_1), p_{\max}(a_1))$. In particular, for $a_1 = -0.9$, we have $p_1(a_1) \approx 0.0431$ and $p_{\max}(a_1) \approx 0.0710$. This explains the appreciable increase in the values of the distribution function $\hat{f}(\xi, p)$ at the points $(\xi, p_{\max}(\xi))$, the displacement of $p_{\max}(\xi)$ to $p = p_1(a_1)$ as $\xi \to a_1$ (see Fig. 4), and the increase in the bubble concentration n in the simple-wave region with decrease in the variable ξ.

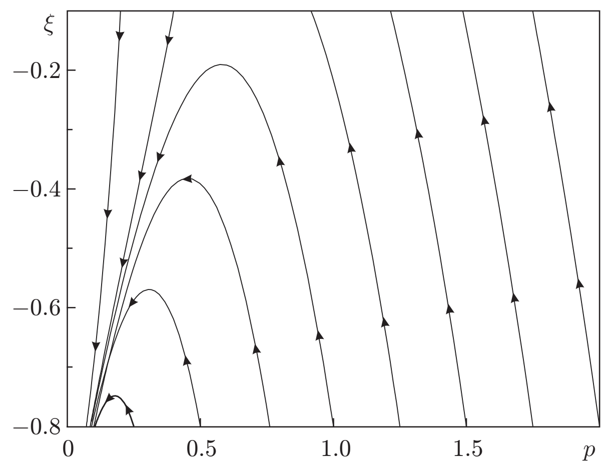

Fig. 5

Conclusions. A method for constructing special classes of exact solutions was proposed for integrodifferential models that admit a formulation in terms of Riemann invariants. Seeking solutions from the special class is considerably simplified compared to the general case and reduces to integrating systems of differential equations. This method was used to obtain a new wide class of solutions for the kinetic model of a bubbly fluid proposed by Russo and Smereka. The solutions of the special class are specified by a system of two differential equations dependent on two arbitrary parameters. In the special class, simple waves are studied and features of the solutions are analyzed. The case of a self-similar simple wave is considered in greater detail.

This work was supported by the INTAS (Grant No. 01-868), the Russian Foundation for Basic Research (Grant No.04-01-00253), and the Foundation "Leading Scientific Schools of Russia" (Grant No. NSh.-440.2003.1).

REFERENCES

1. A. Biesheuvel and L. van Wijangaarden, "Two-phase flow equations for a dilute dispersion of gas bubbles in liquid," *J. Liquid Mech.*, **148**, 301–318 (1984).
2. A. Biesheuvel and W. C. M. Gorissen, "Void fraction disturbance in a uniform bubble liquid," *J. Multiphase Flow*, **16**, 211–231 (1990).
3. G. Russo and P. Smereka, "Kinetic theory for bubble flow I: Collisionless case," *SIAM J. Appl. Math.*, **56**, No. 2, 327–357 (1996).
4. V. M. Teshukov, "Kinetic model of bubbly flow," *J. Appl. Mech Tech. Phys.*, **41**, No. 5, 879–888 (2000).
5. V. M. Teshukov and S. L. Gavrilyuk, "Kinetic model for the motion of compressible bubbles in perfect fluid," *Europ. J. Mech. B, Liquids*, **21**, 469–491 (2002).
6. L. M. Milne-Thomson, *Theoretical Hydrodynamics*,Macmillan, London (1960).
7. V. M. Teshukov, "Characteristics, conservation laws, and symmetries of kinetic equations of motion of bubbles in a fluid," *J. Appl. Mech. Tech. Phys.*, **40**, No. 2, 263–277 (1999).
8. A. A. Chesnokov, "Exact solutions of the one-dimensional Russo–Smereka kinetic equation," *J. Appl. Mech. Tech. Phys.*, **41**, No. 4, 593–804 (2000).
9. V. M. Teshukov, "On the kinetic modeling of the concentration waves in dilute bubbly flows," in: *Proc. of the 2nd Int. Symp. on Two-Phase Flow Modeling and Experimentation* (Rome, Italy, 23–26 May, 1999), Edizioni ETS, Pisa (1999), pp. 1101–1107.
10. S. Prösdorf, *Einige Klassen Singularer Gleichugen*, Alkademie Verlag, Berlin (1977).
11. F. D. Gakhov, *Boundary-Value Problems* [in Russian], Nauka, Moscow (1977).
12. I. I. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York (1980).